Effect of vertical vibrations on avalanches in granular systems

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(Received 24 March 1994)

We discuss the effect of vertical vibration on the dynamics of avalanches on the surface of quasi-two-dimensional granular systems using a minimal "mesoscopic" model in a regime where no bulk convection of the grains is present. We give a simple picture of why large vibration strengths or high vibration frequencies can drastically reduce the static limit of the dynamic friction coefficient entering the minimal model. We derive an exact relation for the minimal angle of repose and perform an asymptotic analysis to obtain the time evolution of the angle of the free surface, which compares well with a recent experiment.

PACS number(s): 05.40.+j, 46.10.+z, 81.35.+k

I. INTRODUCTION

Granular systems (such as sand dunes or dry coffee grounds in a filter) are part of our everyday lives. They show many peculiar properties which do not fit under the standard division of states of matter, into solids, fluids, and gases [1]. Considering a granular system as a classical many-particle system, an *ab-initio*-theory is a difficult task. It involves the collective dynamics of inhomogeneously distributed, finitely extended grains of complicated shapes which interact via static and dynamic friction and inelastic collisions in the presence of gravity.

Of particular scientific *and* technological interest is the response of granular systems to external vibrations, which can dilatate the grains and therefore fluidize the system. External vibrations act as a driving mechanism to generate flow and lead to a variety of phenomena, such as avalanches in sandpiles [2–4], pattern formation [5], surface waves [6], and size segregation [7] ("Brazil nut effect" [8]). There are several numerical studies [9] based on two-dimensional hard disk models; the theoretical understanding of granular systems and their transition to flow, however, still presents a challenge. There is no statistical or "hydrodynamic" theory of granular flow available yet which allows the interpretation of these phenomena.

In order to gain insight into the physics of onset to flow, several important, highly idealized experiments [2,3] have been reported recently. They deal with the dynamics of avalanches on the surface of sandpiles in drums or rotating cylinders. Without vibrations, an avalanche can start if the angle φ of the free surface is larger than a maximum angle of repose φ_s. It comes to a stop again if the angle reaches a minimum angle of repose φ_r. Adding vertical, high-frequency vibrations to a granular pile in the drum geometry, Jaeger, Liu, and Nagel [2] found experimentally a topological change in the dynamics: The angle of the pile relaxes *logarithmically* over a large interval in time to values close to zero.

Recently, we have proposed a minimal "mesoscopic" model [10,11] for the dynamics of avalanches in granular systems. This model applies to situations when the dynamics of the system is concentrated to a narrow layer at the surface with no global convection in the bulk taking place. It consists of two coupled mean field equations for the velocity $v(t)$ of a grain at the surface and the angle $\varphi(t)$ of the surface of the pile, which is quite similar to Coulomb's theory of friction of a body on an inclined plane. Note that $v(t)$ and $\varphi(t)$ must be considered as averages over a large number of experimental runs to level out the details of one individual avalanche. In the special case of no external rotation, it reads

$$\dot{v}=g[\sin\varphi-(b_0+b_2v^2)\cos\varphi]\chi(\varphi,v) \ , \tag{1a}$$

$$\dot{\varphi}=-av \ , \tag{1b}$$

with $\chi(\varphi,v)=\Theta(v)+\Theta(\varphi-\varphi_s)-\Theta(v)\Theta(\varphi-\varphi_s)$ and $\Theta(y)=0$ (1) if $y\leq 0$ ($y>0$). Here, g is the gravitational acceleration and a the coupling coefficient between angle and velocity. The function χ accounts for the static friction (cf. also Ref. [3]): The pile is at rest if $v=0$ *and* $\varphi\leq\varphi_s$, and it is trapped at the rest state if v goes to zero *and* $\varphi\leq\varphi_s$; otherwise, the system evolves dynamically. Dynamic friction is modeled by a friction coefficient $k_d(v)=b_0+b_2v^2$, with b_0 and b_2 both positive [11]. The static limit of the friction coefficient $k_d(v\to 0)$ corresponds to an angle $\varphi_d=\arctan(b_0)$, which is the *basic quantity* in the following.

At first sight, one could think that vertical vibrations are reflected in (1) by modulation of the gravity acceleration g. Our numerical studies, however, have shown that this cannot explain the experiments [2]. In Ref. [10] we have briefly mentioned that the logarithmic relaxation of $\varphi(t)$, found experimentally by Jaeger, Liu, and Nagel [2], can be explained as follows: Assume that the leading order effect of vertical vibrations implies a decrease for φ_d (without vibrations a quantity of order unity) to an angle φ_d^v which is close to zero.

In this paper we substantiate this idea in detail. The paper is divided in two parts: In Sec. II we discuss, on the basis of a simple physical one-grain picture, why vibrations might strongly reduce friction. In Sec. III we work out the effect of reduced kinematic friction using a systematic expansion of (1) around φ_d^v. We derive an expression for the minimal angle of repose φ_r (the long-time limit) for the model as a function of φ_d^v. Moreover, we study the transient dynamics of $\varphi(t)$ in the limit of small φ_d^v and find a transient $\ln(t)$-relaxation for φ.

II. CHANGE OF φ_d DUE TO VERTICAL VIBRATIONS

Here, we want to demonstrate that vertical vibrations can reduce the magnitude of the static limit of the friction coefficient $b_0 = k_d(v \to 0)$. Vibration dilatates the surface layer of the pile, and therefore the grains undergo nonfrictional motion for part of the time. This in turn reduces the effective friction in our model (1). Although this seems to be intuitively clear, a mathematical description is rather difficult. To gain insight into this, we consider a simple, very restricted mechanical one-grain model that is a variant of the inelastic bouncing ball problem [13]. Suppose a single (pointlike) grain is situated on a rough surface, which oscillates vertically with a vibration amplitude Δ and a vibration frequency ω. Scaling time by $t \to \tau = \omega t$, the position of the surface is given by $y_s(\tau) = \Delta \cos\tau$. Suppose further that the grain lies on the surface only because of its weight (no other interactions) and that possible collisions of the grain with the rough surface are completely inelastic. At $\tau = 0$, the surface and the grain are in their maximal position. Depending on the strength and frequency of the vibrating surface, two types of dynamics for the grain are possible: (i) the grain stays forever on the surface, i.e., $y_g(\tau) = \Delta \cos\tau$ or (ii) it undergoes a free fall due to gravity until it reaches the surface again, i.e., $y_g(\tau) = \Delta - (g/2\omega^2)\tau^2$. Then, the grain stays at the surface [the constraint is $y_s(\tau) \le y_g(\tau)$] until the maximum $y_s = y_g = \Delta$ is reached again. The time τ_f, when the grain hits the oscillating surface again, is given by the solution of

$$\cos\tau_f = 1 - \tfrac{1}{2}\beta\tau_f^2 \ , \tag{2}$$

with $\beta = g/\Delta\omega^2$. Therefore, large Δ and/or large ω imply *small* β. From (2) one immediately infers that a part-time free fall can only occur if $\beta < 1$. For small β, one obtains $\tau_f \simeq 2\pi(1-\sqrt{\beta})$. In the limit $\beta \to 0$, $\tau_f = 2\pi$. During the interval $[0,\tau_f]$ the grain undergoes a free fall and therefore moves basically frictionless (we neglect here friction due to the surrounding air), while during $[\tau_f, 2\pi]$ the grain undergoes friction as in the nonvibrating case. The effective static limit of the dynamic friction coefficient b_0^{eff} in this model is therefore given by the average over one period,

$$b_0^{\text{eff}} = b_0(1 - \tau_f/2\pi) \ . \tag{3}$$

In Fig. 1 we show the dependence of the ratio b_0^{eff}/b_0 as a function of $\beta/2$. This ratio approaches zero for small β, increases almost linearly for $0.2 < \beta < 0.8$, and increases for $0.8 < \beta \le 1$ stronger than linear to unity. Above $\beta = 1$, b_0^{eff}/b_0 equals unity in this simple model. The angle φ_d^v corresponding to b_0^{eff} is given by $\varphi_d^v = \arctan(b_0^{\text{eff}})$. For small $\beta = g/\Delta\omega^2$, one finds

$$\varphi_d^v = \arctan\left[\sqrt{g/\Delta\omega^2}\, b_0\right] \simeq \sqrt{g/\Delta\omega^2}\tan\varphi_d \ . \tag{4}$$

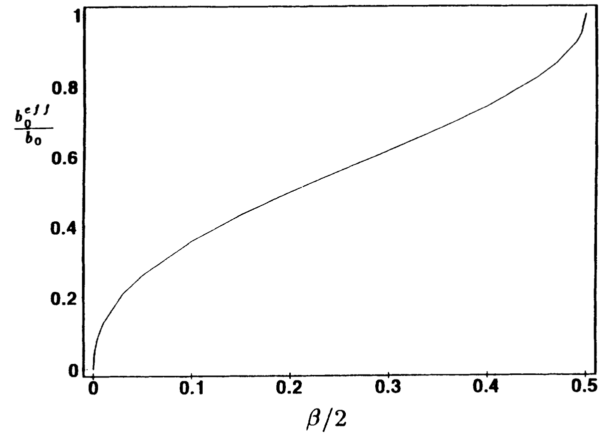

FIG. 1. Changes of the static limit of the friction coefficient due to vibration: Ratio b_0^{eff}/b_0 as a function of $\beta/2 = g/2\Delta\omega^2$, cf. Eq. (3).

From (4), we see that in the limit of large vibration strength and/or high vibration frequency, φ_d^v *approaches zero,* proportional to ω^{-1} and $\Delta^{-1/2}$. This supports our earlier speculation in Ref. [10] on the effect of vibrations on b_0 in granular systems.

In this simple picture, changes of b_0 can only appear if $\beta < 1$. This is caused by the very restrictive assumptions we use. In general, one has also to take into account partly nonvertical and partly elastic bouncing of the grains, as well as collision effects. This might reduce the size of b_0 considerably, even if $\beta > 1$, as is likely the case in the experiment of Jaeger, Liu, and Nagel [2]. Also, other mechanisms are reasonable, e.g., successive destabilization of neighboring grains (domino effect). We note that often bulk convection occurs in experiments [1] if $\beta < 1$, which limits the applicability of the above scenario. Nevertheless, it shows that dilatation of the surface layer due to external vibration reduces the friction of an individual grain at the surface.

III. MINIMAL MODEL IN THE PRESENCE OF VERTICAL VIBRATIONS

To investigate the dynamics of avalanches moving down a pile of grains under the influence of vertical vibrations, we go back to the mesoscopic model (1). First, Benza, Nori, and Pla (cf. Ref. [13]) have shown that the coupling parameter a in the drum geometry can be written as $a = \hat{a}\tan\varphi$, with $\hat{a}$ independent of φ. Therefore we can approximate a by $\hat{a}\tan\varphi_d^v$. Second, we suppose that φ_d^v is drastically reduced by the external vibrations. We emphasize that the discussion in this section *only uses the fact that* $\varphi_d^v \ll 1$, *independent of the physical scenario how this can be explained.* For the following, it proves useful to introduce

$$\epsilon = \sin\varphi_d^v/(1-\sin^2\varphi_d^v) \ , \tag{5}$$

implying that ϵ is positive. Restricting the following to $v > 0$, i.e., $\chi = 1$ in (1), combining (1a) and (1b) to a single equation in φ, introducing the scaled deviation from φ_d^v,

$$\Psi(t) = [\varphi(t) - \varphi_d^v]/\epsilon \ , \tag{6}$$

expanding the differential equation in powers of $\Psi(t)$, and retaining only terms up to quadratic powers in Ψ, leads to

$$\ddot{\Psi}-n\dot{\Psi}^2+\epsilon\Psi=0 , \tag{7}$$

where $n=gb_2/\hat{a}>0$. Note that all information on the dependence on φ_d^v is now contained in ϵ. The velocity v is related to the dynamics of Ψ by $v=-\epsilon\dot{\Psi}$. The constraint for the dynamics, $v>0$, then reads $\dot{\Psi}<0$. Equation (7) describes the evolution of the angle of the surface of the pile until $v(t)=0=-\epsilon\dot{\Psi}(t)$ is reached. Then the avalanche stops, i.e., $v(t)=0$, and the angle $\Psi(t)$ remains constant in time given by the minimal angle of repose $\Psi_r=(\varphi_r-\varphi_d^v)/\epsilon$. The effect of strong vertical vibrations is given by the limit of very small "frequencies" $\sqrt{\epsilon}$ in the oscillator with quadratic friction, Eq. (7), since $\epsilon\simeq\varphi_d^v$.

A. Constant of motion and minimal angle of repose

Equation (7) has an important and surprising property: It is *integrable*. The constant of motion J can be calculated explicitly, reading

$$J=\left[\frac{1}{2}\dot{\Psi}^2-\frac{\epsilon}{4n^2}(1+2n\Psi)\right]e^{-2n\Psi} . \tag{8}$$

It is trivial to check that $dJ/dt\equiv 0$ is fulfilled. A short, *constructive* proof of (8) goes as follows: First, add a term $-\epsilon/2n+\epsilon/2n=0$ in (7), multiply the resulting equation by $\dot{\Psi}\exp(-2n\Psi)$, and finally rearrange the terms in a way that a time derivative can be put outside of the brackets to obtain (8). In the Appendix, we discuss additional mathematical properties of (7) [12].

The first integral J is determined by the relevant initial conditions $\Psi_s=\Psi(0)=(\varphi_s-\varphi_d^v)/\epsilon$ and $\dot{\Psi}(0)=-\kappa/\epsilon$, with κ being the initial velocity, $\kappa\equiv v(0)$. Therefore, we obtain

$$J=\left[\frac{\kappa^2}{2\epsilon^2}-\frac{\epsilon}{4n^2}(1+2n\Psi_s)\right]e^{-2n\Psi_s}=\text{const} , \tag{9}$$

as long as the avalanche moves. We can take advantage of (8) and (9) to derive a relation for the minimum angle of repose $\Psi_r=(\varphi_r-\varphi_d^v)/\epsilon$, given by the condition that the velocity v equals zero, and setting therefore $\dot{\Psi}=0$. From (8) one obtains the condition

$$\exp(2n\Psi_r)=-\epsilon(1+2n\Psi_r)/4n^2J . \tag{10}$$

This condition can be easily solved leading to an *exact* expression for the minimal angle of repose in (7),

$$\Psi_r=\frac{1}{\epsilon}(\varphi_r-\varphi_d^v)=-\frac{1}{2n}\left[W\left[\frac{4n^2J}{\epsilon e}\right]+1\right] . \tag{11}$$

Here, $W(x)$ is Lambert's function, i.e., the solution of the functional relation $W(x)\exp[W(x)]=x$. A numerical plot of $W(x)$ is shown in Fig. 2(a). We note two properties of $W(x)$. (i) A solution of the functional equation for $W(x)$ only exists if $x>x_{\min}=1/e\simeq-0.36788$. Therefore, a minimal angle of repose exists if the initial conditions $\Psi(0)$ and $\dot{\Psi}(0)$, as well as the model parameters n and ϵ, fulfill $4n^2J/\epsilon e>x_{\min}$. This is fulfilled for the parameters we use in the following. (ii) For small moduli of x, $W(x)=x-x^2+O(x^3)$ holds. Naturally, (11) can also be used to determine φ_r in the case *without* vertical vibrations by substituting φ_d^v by φ_d.

Small ϵ and zero initial velocity $\kappa=0$ imply for the constant of motion, $J\sim(\varphi_s-\varphi_d^v)\exp[-2n(\varphi_s-\varphi_d^v)/\epsilon]$. For small ϵ and *nonzero* initial velocity $\kappa\neq 0$, one obtains $J\sim(\kappa^2/2\epsilon^2)\exp[-2n(\varphi_s-\varphi_d^v)/\epsilon]$. In both cases, the exponential factor in J leads to very small moduli of J and J/ϵ for small ϵ. Therefore, the minimal angle of repose φ_r approaches φ_d^v linearly in ϵ as $\epsilon\to 0$ according to

$$\varphi_r-\varphi_d^v\simeq-\frac{\epsilon}{2n} , \tag{12}$$

which is independent of the initial velocity κ. The minimum angle of repose, φ_r is equal to φ_d^v only if ϵ equals zero. For small positive ϵ, $\varphi_r<\varphi_d^v$ holds.

In Fig. 2(b) we show the dependence of $\varphi_r-\varphi_d^v$ for $0\leq\epsilon\leq 0.3$ and for different initial velocities $\kappa=v(0)=0$, 0.01, 0.02, and 0.03 and $n=1$. As discussed above, all four curves begin at $\epsilon=0$ and $\varphi_r-\varphi_d^v=0$. For *zero* initial velocity [cf. curve (α) in Fig. 2(b)], $\varphi_r-\varphi_d^v$ decays monotonically to a limit value $-(\varphi_s-\varphi_d^v)$, corresponding to $\varphi_r=2\varphi_d^v-\varphi_s$. For *nonzero* initial velocities [curves (β),

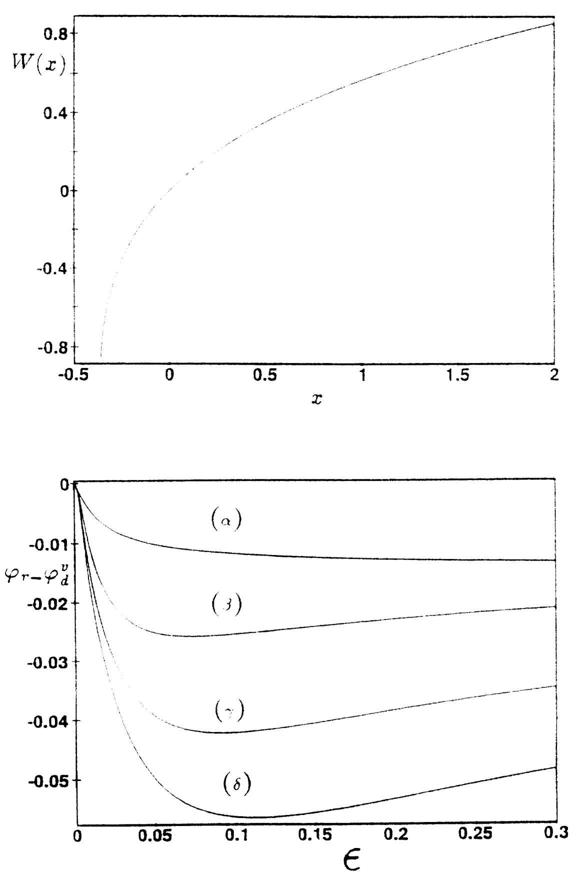

FIG. 2. (a) Solution $W(x)$ of the functional relation $W(x)\exp[W(x)]=x$. (b) Dependence of $\varphi_r-\varphi_d^v$ on ϵ for a fixed $n=1$, $\varphi_s=0.48$, and initial velocities (α) $\kappa=0$, (β) $\kappa=0.01$, (γ) $\kappa=0.02$, and (δ) $\kappa=0.03$.

(γ), and (δ) in Fig. 2(b)], the decay of $\varphi_r - \varphi_d^v$ is linear for $\epsilon < 0.002$. Above this value, $\varphi_r - \varphi_d^v$ decays strongly to reach a minimum and subsequently slowly increases again. The larger the initial velocity, the more pronounced is this behavior.

B. Perturbation theory for small ϵ

To investigate the *transient* dynamics of avalanches when they approach the minimum angle of repose φ_r given by (11), we perform a perturbation analysis of (7) for small ϵ, i.e., small φ_d^v. Inserting an expansion of $\Psi(t)$ in powers of ϵ,

$$\Psi(t) = \Psi_0(t) + \epsilon\Psi_1(t) + O(\epsilon^2) , \tag{13}$$

in (4), the first two orders read

$$\ddot{\Psi}_0 = n\dot{\Psi}_0^2 , \tag{14a}$$

$$\ddot{\Psi}_1 = 2n\dot{\Psi}_0\dot{\Psi}_1 - \Psi_0 . \tag{14b}$$

The solutions corresponding to the initial conditions $\Psi_0(0) = \Psi_{ss}$, $\dot{\Psi}_0(0) = -\kappa/\epsilon$, $\Psi_1(0) = 0$, and $\dot{\Psi}_1(0) = 0$ read

$$\Psi_0(t) = \Psi_{ss} - \frac{1}{n}\ln(1 + n\kappa t/\epsilon) , \tag{15a}$$

$$\Psi_1(t) = \frac{1}{6n^3\kappa^2}(\epsilon + n\kappa t)^2 \ln(1 + n\kappa t/\epsilon) + f(t) , \tag{15b}$$

$$f(t) = -\frac{6\epsilon^2\kappa n t + (18n\Psi_{ss} + 15)\epsilon(\kappa n t)^2 + (\kappa n t)^3(6n\Psi_{ss} + 5)}{36n^3\kappa^2(\epsilon + n\kappa t)} . \tag{15c}$$

Note that the perturbation expansion (13,15) is nonuniform in time and only valid as long as $t < O(1/\epsilon)$. This, however, is sufficient to discuss the transient relaxation. As an initial angle we used here Ψ_{ss}, since in the presence of vibrations the pile can be unstable even at angles smaller than Ψ_s [2], the maximum angle of repose without vibrations.

The lowest-order solution, Eq. (15a), is of the same form as the decay law of Jaeger, Liu, and Nagel [2] found by fitting their experimental data. Note that (15a) decays monotonically. Incorporating the next order, Eq. (15b), shows that Ψ must have a minimum at finite times since Ψ_1 diverges, proportional to $t^2 \ln t$, in the long-time limit. A minimum of Ψ, however, implies $v = 0$ and thus a halt of the avalanche. This minimum is also an indication of the breakdown of the perturbation expansion.

In Fig. 3 we show as a representative example plots of $\epsilon\Psi_0$ and $\epsilon\Psi$ as functions of $\ln t$. One can see that Ψ_0 as well as Ψ decay logarithmically over a wide range of time. Ψ_0 and Ψ agree within a linewidth for $\ln t < 2$, implying that already Ψ_0 is a reasonable approximation for not too large times. The numerical solution shows similar behavior; it saturates, however, in the minimal angle of repose $\varphi_r = \varphi_r(\Psi_{ss}, \kappa, \epsilon, n)$ given by Eq. (11). In general, the minimum angle of repose is nonzero; for small φ_d^v, however, the minimum angle is close to zero, corresponding to an almost horizontal surface of the pile.

As a consequence of (13) and (15), one obtains in lowest-order approximation that the velocity decays algebraically in time, i.e.,

$$v(t) = -\epsilon\dot{\Psi} \simeq -\epsilon\dot{\Psi}_0 = \frac{\kappa}{1 + \kappa n t/\epsilon} = \frac{v(0)}{1 + v(0)nt/\epsilon} . \tag{16}$$

If $\epsilon \ll v(0)nt$, then $v(t) \simeq (\epsilon/n)t^{-1}$ holds. This might serve as a simple way to determine the ratio ϵ/n of the model experimentally.

C. The role of nonzero initial velocity due to vibrations

From (15a) one can see that a nonzero initial velocity $v(0) = \kappa$ is required to obtain the logarithmic decay of Ψ in t. A physical explanation of that goes as follows: Turning on vibration jump starts the evolution of avalanches already at angles where, without vibration, the inclined pile is stable. In a more microscopic picture, vibration induces dilatation of the grains at the surface of the pile, leading to a destabilization of the system. This is reflected in our model by a characteristic initial velocity $v(0) = \kappa$. $v(0)$ is basically a function of the vibration amplitude and frequency [and also of other control parameters like n, ϵ, and the initial angle $\Psi(0)$]; it is, however, characteristic of a given experimental setup.

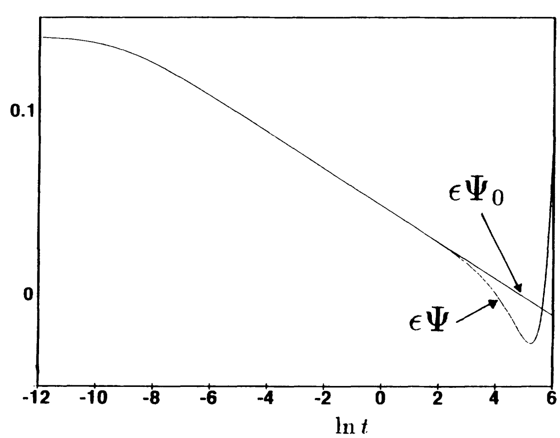

FIG. 3. Transient evolution of the angle on a logarithmic time scale. Parameters are $\epsilon = 0.001$, $n = 1$, $\Psi_s = 0.14/\epsilon$, and $\kappa = 10^5\epsilon$. The curves represent the lowest order approximation $\epsilon\Psi_0$, Eq. (15a), and the next higher approximation $\epsilon\Psi \simeq \epsilon(\Psi_0 + \epsilon\Psi_1)$, Eq. (15). They agree within linewidth for $\ln t < 2$.

IV. CONCLUSION AND DISCUSSION

We have studied the impact of vertical vibrations in a one-dimensional model for avalanches on the surface of granular piles. Our focus is the mathematical description of a quasi-two-dimensional system like the drum geometry. A simple inelastic bouncing ball argument has been given to explain why the static limit of the friction coefficient might be drastically diminished. This argument is idealized; it does not take into account partly elastic bouncing of the grains as well as possible collisions of the grains. These effects can change the value where deviations of b_{eff} from b_0 become significant. We have calculated an exact expression for the minimal angle of repose, φ_r of (7), i.e., the angle where the system comes to rest again. Finally, we have studied the transient approach to the minimal angle of repose using perturbation methods. We could recover the logarithmic decay law for $\varphi(t)$ [2] whenever vibration-induced reduction of the static limit of the friction coefficient b_0 occurs, i.e., $\epsilon \ll 1$. There are three different prior explanations of the logarithmic decay law: Jaeger, Liu, and Nagel [2] have drawn an analogy to activated hopping over a barrier where vibrations mimic an effective temperature. Duke, Barker, and Mehta [14], using Monte Carlo simulations, attributed it to the cumulative effect of small changes in the network of contacts within the pile (cf. also Ref. [1]). Finally, Mehta, Needs, and Dattagupta [15] have discussed a coupled set of Langevin equations for the macroscopic angle of tilt and its local variation. In their statistical treatment they found within some limits an almost logarithmic relaxation in time.

Our present argument is purely mechanical; the ingredients are a strongly reduced dynamic friction due to external vibration and a nonzero initial velocity produced by vibrations. Otherwise, our approach allows for a deterministic description of the dynamics of avalanches under vertical vibration. We hope that our study stimulates further experiments on "mesoscopic" modeling of avalanches.

ACKNOWLEDGMENTS

We thank U. Eckern for discussions on the mechanical interpretation of our model and H. Jaeger for comments on the manuscript.

APPENDIX

Here, we discuss briefly, the mathematical structure of the (nonlinear) oscillator with quadratic friction,

$$\ddot{\Psi}(t)-n\dot{\Psi}(t)^2+\epsilon\Psi(t)=0\ , \tag{A1}$$

discarding the additional restriction $\dot{\Psi}<0$ required in our physical problem. Obviously, (A1) possesses *invariance under time reversal,* $t \to -t$ being the underlying reason for the existence of the constant of motion J, Eq. (8). Therefore, one can interpret J as a (nondimensionalized) energy, $E=J=T(\dot{\Psi},\Psi)+V(\Psi)$, with

$$T(\dot{\Psi},\Psi)=\tfrac{1}{2}e^{-2n\Psi}\dot{\Psi}^2,\quad V(\Psi)=-\frac{\epsilon}{4n^2}(1+2n\Psi)e^{-2n\Psi} \tag{A2}$$

being the kinetic and potential energy, respectively. This implies that (A1) can be derived from Hamilton's principle using the Lagrangian function $L(\Psi,\dot{\Psi})=T(\dot{\Psi},\Psi)-V(\Psi)$. One can easily check that $L(\Psi,\dot{\Psi})$ fulfills indeed the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\Psi}}-\frac{\partial L}{\partial\Psi}=0\ .$$

The canonically conjugate momentum to Ψ is given by

$$\Pi=\frac{\partial L}{\partial\dot{\Psi}}=e^{-2n\Psi}\dot{\Psi}\ , \tag{A3}$$

implying that the Hamiltonian function belonging to Eq. (A1) reads

$$H(\Psi,\Pi)=\tfrac{1}{2}e^{2n\Psi}\Pi^2+V(\Psi) \tag{A4}$$

and fulfills the Hamiltonian equations $\dot{\Psi}=\partial H/\partial\Pi$ and $\dot{\Pi}=-\partial H/\partial\Psi$. Using (A2) one can interpret (A1) as the motion of an undamped pendulum with angle-dependent "mass" in a nonsymmetric potential $V(\Psi)$. If $\epsilon>0$ ($\epsilon<0$), $V(\Psi)$ has a minimum (maximum) at $\Psi=0$ with $V(0)=-\epsilon/4n^2$. For small Ψ one obtains a quadratic potential $V(\Psi)\simeq-(\epsilon/4n^2)(1-2n^2\Psi^2)$, with ϵ being the curvature. Therefore, small ϵ corresponds to a flat potential $V(\Psi)$ close to the origin $\Psi=0$. We note that (A1) can have two different types of dynamics for positive ϵ. If $-1<4n^2E/\epsilon<0$ holds, the system shows periodic solution; if $4n^2E/\epsilon<0$, the solutions diverge in time.

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[11] This model is minimal in the sense that its friction coefficient $k_d(v)$ is the simplest to fulfill the physical limits, in particular the quadratic increase with v for large v. In particular, it differs from related models [cf. H. M. Jaeger, C. Liu, S. R. Nagel, and T.A. Witten, Europhys. Lett. **18**, 619 (1990), V. G. Benza, F. Nori, and O. Pla, Phys. Rev. E **48**, 4095 (1993), and also the model in Ref. [3]] by assuming that $k_d(v)$ increases monotonically with v.

[12] As in the case of a pendulum, one can reduce Eq. (7) to a first order equation in Ψ,

$$\dot{\Psi} = \pm\sqrt{2J\exp(2n\Psi)+(\epsilon/2n^2)(1+2n\Psi)}\ .$$

Its formal, implicit solution reads

$$\mp\sqrt{2}n\int^{\Psi(t)} dy\,[\epsilon(1+2ny)+4n^2J\exp(2ny)]^{-1/2}+t=\text{const.}$$

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